

Precise constraints on a τ function in 2D quantum gravity

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Abstract

For an arbitrary p , propose a new and computable method which can determine the values of unknown constants in constraints on a tau function which satisfies both the p-reduced KP hierarchy and the string equation. All the constants do not equal 0, unlike what people usually think of. With these values, obtain the precise algebra that the constraints compose. This algebra includes none of $\{t_{mp}\}$ and also includes the Virasoro algebra as a subalgebra.

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§1. Introduction

Quantum gravity is an interesting object in the current research of mathematical physics. In 2D quantum gravity, Kontsevich[1] proved Witten's conjecture[2] that two different approaches to 2D quantum gravity coincide. That is, a partition function for the intersection theory of moduli space is the logarithm of some τ function[3] which satisfies the string equation and the 2-reduced KP hierarchy. Meanwhile, using Kontsevich's matrix integral representation of the partition function, Witten[4] showed the exponent of the partition function is a vacuum vector for the Virasoro algebra. Together with the conclusion[5, 6] that a τ function which satisfies the string equation and the 2-reduced KP hierarchy is equivalent to a vacuum vector for the Virasoro algebra, he also obtained the equivalence of the two approaches. Since integrable system has close connection with the string theory and the intersection theory[7, 8], this conclusion has been widely researched with various methods in this field[9, 10, 11, 12, 13]. An interesting problem is to extend the conclusion in [5, 6] from 2 to an arbitrary p , which is to obtain the equivalence between a τ function constrained by the string equation and the p -reduced KP hierarchy and a vacuum vector of some algebra which include the Virasoro algebra as a subalgebra. When $p = 3$, Goeree[14] showed that it is true. And the case for bigger p had also been researched in [6, 7, 13].

In order to obtain the above equivalence for an arbitrary p , it need to obtain the precise constraints which the KP hierarchy and the string equation impose on tau functions. When we calculate them, it creates a lots of constants in the obtained constraints whose values are unknown. When $p = 2$, the constrains constitute the Virasoro algebra and we could use the commutation relations of the Virasoro algebra to calculate the values of the constants. But when $p \geq 4$, although there are some classical conclusions about W algebra[6, 13], it is so hard to calculate the commutation relations of the constraints that the constants in the higher order constraints are still unknown. As far as we have known, there is not an effective computable method to determine them when $p \geq 4$. Due to the uncertainty of the constants, we also could not obtain the precise algebraic structure of these constraints. In this study, for an arbitrary p , we propose a new computable method which can determine values of the constants. It is a recursive process and we can directly calculate them step by step. It is usually that assign the value of 0 to all the constants; but here, by this method, we know that all of them are not equal 0. And the none zero constants are closely related with the centers of the algebra that the constraints constitute. When $p = 2$, our conclusion coincide with the current conclusion, that is, the constants determined through our method being the same as those determined through commutation relations of Virasoro. Consequently, with these values we obtain the precise constraints. And further we obtain the precise algebra which the constraints constitute. The algebra does not include the redundant variables of t_{mp} , and we find the connection between the algebra and the $W_{1+\infty}$ algebra which include all the variables of t_{mp} . Based on this connection, we can calculate the commutation relations of one algebra from those of the other algebra. And the calculation is much simpler than a straightforward calculation. Furthermore, the obtained algebra include the Virasoro algebra as a subalgebra. So the above conclusion in [5, 6] is also included in our conclusion. In addition, we mainly use the tool of pseudo-differential operators to prove the conclusions, which is introduced by Dickey[15] and greatly simplifies the proof.

The organization of the paper is as follows. In section 2, for self-contained we give a brief

description of the KP hierarchy. In section 3, we prove the connection between $W_{1+\infty}$ algebra and the algebra of $\bar{W} = \{W_n^{(m)}|_{t_{mp}=0}\}$. In section 4, we show the approach to calculate unknown constants, which is our main theorem. Meanwhile, we give some examples for our theorems. Section 5 is devoted to conclusions.

§2. KP hierarchy

To be self-contained, we give a brief introduction to the KP hierarchy based on a detailed research in [15].

Let F be an associative ring of functions which include infinite time variables $t_i \in \mathbb{R}$:

$$F = \{f(t) = f(t_1, t_2, \dots, t_j, \dots); t_i \in \mathbb{R}\}.$$

Denote ∂_{t_1} by ∂ , which is the common differential operator on the first variable t_1 . Its actions on $f(t)$ are

$$\partial f(t) = \partial_{t_1} f(t); \quad \partial \circ f(t) = f(t) \partial + \partial_{t_1} f(t). \quad (2.1)$$

Here the symbol “ \circ ” denote the multiplication between operators. If we consider a function $f(t)$ as a operator whose action on $g(t) \in F$ is $f(t)g(t)$, we can infer the following identity about multiplication of function operators and differential operators. That is for any $j \in \mathbb{Z}$

$$\partial^j \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\partial^i f) \partial^{j-i}, \quad \binom{j}{i} = \frac{j(j-1) \cdots (j-i+1)}{i!}, \quad j \in \mathbb{Z}. \quad (2.2)$$

If the function operators are located on the the left-hand side we omit “ \circ ”. So with (2.2) we could obtain an associative ring $F(\partial)$ of formal pseudo differential operators, which includes two operations “ $+$ ” and “ \circ ”:

$$F(\partial) = \left\{ R = \sum_{j=-\infty}^d f_j(t) \partial^j, f_j(t) \in F \right\}.$$

The ring $F(\partial)$ includes two subrings which are $F_+(\partial) = \{R_+ = \sum_{j=0}^d f_j(t) \partial^j\}$, the ring of deference operators and $F_-(\partial) = \{R_- = \sum_{j=-\infty}^{-1} f_j(t) \partial^j\}$, the the ring of Volterra operators.

Let L be a general first order pseudo differential operator:

$$L = \partial + \sum_{j=1}^{\infty} f_j(t) \partial^{-j}. \quad (2.3)$$

The KP-hierarchy[15] can be expressed as

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L]. \quad (2.4)$$

Comparing the powers of ∂ on both sides, we can obtain the family of equations in functions $f_j(t)$. From $i = 2$ and $i = 3$, we obtain

$$3\partial_{t_2}^2 f = \partial_{t_1}(4\partial_{t_3} f - \partial_{t_1}^3 f - 6f\partial_{t_1} f), \quad (2.5)$$

which is the famous Kadomtsev-Petviashvili equation.

Define the dressing operator

$$\Phi(t) = 1 + \sum_{j=1}^{\infty} w_j(t) \partial^{-j},$$

which satisfies

$$L = \Phi(t) \circ \partial \circ \Phi(t)^{-1}. \quad (2.6)$$

Then the Sato equation in the operator $\Phi(t)$

$$\frac{\partial \Phi(t)}{\partial t_i} = -(L^i)_- \circ \Phi(t) \quad (2.7)$$

is equivalent to the KP equation (2.4). The KP hierarchy can also be expressed as the following equations equivalently:

$$L^k w = z^k w \quad \text{and} \quad \partial_{t_m} w = L_+^m w. \quad (2.8)$$

Using the dressing operator Φ , we can give a kind of form solutions for the above equations which are called Baker or wave functions $w(t, z)$ and adjoint Baker or wave functions $w^*(t, z)$:

$$w(t, z) = \Phi(t) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) = \left(1 + \frac{w_1(t)}{z} + \frac{w_2(t)}{z^2} + \cdots\right) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \quad (2.9)$$

and

$$w^*(t, z) = (\Phi^{-1}(t))^* \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right) = \left(1 + \frac{w_1^*(t)}{z} + \frac{w_2^*(t)}{z^2} + \cdots\right) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right). \quad (2.10)$$

a KP hierarchy is equivalent to a single function, the tau function $\tau(t)$, which means that all functions $w_i(t)$ in the dressing operator $\Phi(t)$ can be generated by a single function, the τ function. That is

$$w(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \quad (2.11)$$

and

$$w^*(t, z) = \frac{\tau(t + [z^{-1}])}{\tau(t)} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right), \quad (2.12)$$

where $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \dots)$. And further all function $f_i(t)$ in the operator L can be generated by the tau function too. Here introduce the $G(z)$ operator, whose action on functions is $G(z)f(t) = f(t - [z^{-1}])$.

There are a family of symmetries for KP hierarchy which are called the additional symmetries. We denote the infinitesimal operators for the symmetries by ∂_{ml}^* . Their actions on $\Phi(t)$ are as follow¹

$$\partial_{ml}^* \Phi(t) = -(M^m L^l)_- \circ \Phi(t). \quad (2.13)$$

where

$$M = \Phi(t) \circ \Gamma \circ \Phi(t)^{-1}, \quad (2.14)$$

and $\Gamma = \sum_{i=1}^{\infty} it_i \partial^{i-1}$.

There are another family of symmetries for the KP hierarchy. Their infinitesimal operators are called vertex operators, which are defined as follow

$$X(\lambda, \mu) =: \exp \sum_{i=-\infty}^{\infty} \left(\frac{P_i}{i\lambda^i} - \frac{P_i}{i\mu^i} \right) : \quad (2.15)$$

where

$$P_i = \begin{cases} \partial_i, & i > 0 \\ |i|t_{|i|}, & i \leq 0 \end{cases}$$

Talyor expand the $X(\lambda, \mu)$ in μ at the point of λ , we have

$$X(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{n=-\infty}^{\infty} \lambda^{-n-m} W_n^{(m)}, \quad (2.16)$$

where

$$\sum_{n=-\infty}^{\infty} \lambda^{-n-m} W_n^{(m)} = \partial_{\mu}^m |_{\mu=\lambda} X(\lambda, \mu).$$

By a straightforward computation, the first items of $W_n^{(i)}$ are as follow

$$W_n^{(0)} = \delta_{n,0}, \quad (2.17)$$

$$W_n^{(1)} = P_n, \quad (2.18)$$

$$W_n^{(2)} = \sum_{i+j=n} : P_i P_j : - (n+1)P_n, \quad (2.19)$$

$$W_n^{(3)} = \sum_{i+j+k=n} : P_i P_j P_k : - \frac{3}{2}(n+2) \sum_{i+j=n} : P_i P_j : + (n+1)(n+2)P_n, \quad (2.20)$$

$$W_n^{(4)} = P_n^{(4)} - 2(n+3)P_n^{(3)} + (2n^2 + 9n + 11)P_n^{(2)} - (n+1)(n+2)(n+3)P_n \quad (2.21)$$

$$\begin{aligned} W_n^{(5)} = & P_n^{(5)} - \frac{5}{2}(n+4)P_n^{(4)} + \left(\frac{10}{3}n^2 + 20n + 35\right)P_n^{(3)} \\ & - \left(\frac{5}{2}n^3 + 20n^2 + \frac{105}{2}n + 50\right)P_n^{(2)} + (n+1)(n+2)(n+3)(n+4)P_n \end{aligned} \quad (2.22)$$

¹Here for convenient the symbol has a slight different from [15].

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where

$$P_n^{(0)} = \delta_{n,0}, \quad (2.23)$$

$$P_n^{(1)} = P_i \quad (2.24)$$

$$P_n^{(2)} = \sum_{i+j=n} : P_i P_j : \quad (2.25)$$

$$P_n^{(3)} = \sum_{i+j+k=n} : P_i P_j P_k : \quad (2.26)$$

$$P_n^{(4)} = \sum_{i+j+k+l=n} : P_i P_j P_k P_l : - \sum_{i+j=n} : ij P_i P_j : \quad (2.27)$$

$$P_n^{(5)} = \sum_{i+j+k+l+m=n} : P_i P_j P_k P_l P_m : - 5 \sum_{i+j+k=n} : ij P_i P_j P_k : \quad (2.28)$$

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The two families of symmetries are equivalent. This fact is represented by the ASvM formula [12, 16, 17]

$$\partial_{m,l+m}^* \tau(t) = \frac{W_l^{(m+1)} \cdot \tau(t)}{m+1}, \quad (2.29)$$

which hold for $m \geq 0$ and for all l .

In integrable system, using the additional symmetries, the string equation [15] can be represent as

$$[L^p, \frac{1}{p}(ML^{-p+1})_+] = 1, \quad (2.30)$$

or equivalently as

$$\partial_{1,-p+1}^* L = 0. \quad (2.31)$$

§3 Constraint equations and connection between two algebras

In this section, firstly we show equations that a tau function, which is under constraints of both the string equation and the p-reduced KP, satisfies. Similar results were obtained by other method. Here we use the method of pseudo-differential operators to obtain them, which simplifies the proof. Secondly, we find the connection between the $W_{1+\infty}$ algebra which includes the redundant variables of $\{t_{mp}\}$ and the algebra of $\bar{W} = \{W_n^{(m)}|_{t_{mp}=0}\}$ which doesn't include $\{t_{mp}\}$. This connection is used in the proof of Theorem 4.8 in next section. Furthermore, using this connection, we could greatly simplify the calculations about commutation relations of \bar{W} .

Now, to obtain the constraint equations for a corresponding τ function, we first show actions of additional symmetries on wave functions.

Lemma 3.1. For any integer $m \geq 0$ and $l \in \mathbb{Z}$,

$$\partial_{m,m+l}^* w(t, z) = \left((G(z) - 1) \frac{W_l^{(m+1)}}{m+1} \cdot \tau(t) \right) \cdot w(t, z). \quad (3.1)$$

Proof: By (2.11) and (2.29),

$$\begin{aligned} & \partial_{m,m+l}^* w(t, z) \\ &= \partial_{m,m+l}^* \left(\frac{G(z)\tau(t)}{\tau(t)} \exp \sum_{i=1}^{\infty} (t_i z^i) \right) \\ &= \frac{\tau(t) \cdot G(z)(\partial_{m,m+l}^* \tau(t)) - G(z)\tau(t) \cdot \partial_{m,m+l}^* \tau(t)}{\tau^2(t)} \exp \sum_{i=1}^{\infty} (t_i z^i) \\ &= \left((G(z) - 1) \frac{W_l^{(m+1)}}{m+1} \cdot \tau(t) \right) \cdot w(t, z). \end{aligned} \quad (3.2)$$

□

Now we consider the string equation. From [13], we know that if the operator L satisfies the p-reduced KP and the string equation, then

$$(M^j L^{kp+j})_- = \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) L^{-p} \quad \text{when } k = -1; \quad j = 1, 2, \dots \quad (3.3)$$

$$= 0 \quad \text{when } k = 0, 1, 2, \dots; \quad j = 1, 2, \dots \quad (3.4)$$

Substitute the above into the definition of the additional symmetry, we have

$$\partial_{j,kp+j}^* \Phi(t) = - \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) L^{-p} \circ \Phi(t) \quad \text{when } k = -1; \quad j = 1, 2, \dots \quad (3.5)$$

$$= 0 \quad \text{when } k = 0, 1, 2, \dots; \quad j = 1, 2, \dots \quad (3.6)$$

Substitute the above into the definition of the wave functions, we have

$$\partial_{j,kp+j}^* w(t, z) = - \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) z^{-p} \quad \text{when } k = -1; \quad j = 1, 2, \dots \quad (3.7)$$

$$= 0 \quad \text{when } k = 0, 1, 2, \dots; \quad j = 1, 2, \dots \quad (3.8)$$

Then, using Lemma 3.1, we could obtain the constraint equations for the τ function which satisfies both the string equation and the p-reduced KP.

Corollary 3.2. If a τ function satisfies both the string equation and the p-reduced KP, then it satisfies

$$(G(z) - 1) \frac{\frac{W_{kp}^{(j+1)}}{j+1} \cdot \tau(t)}{\tau(t)} = - \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) z^{-p} \quad \text{when } k = -1; \quad j = 1, 2, \dots \quad (3.9)$$

$$= 0 \quad \text{when } k = 0, 1, 2, \dots; \quad j = 1, 2, \dots \quad (3.10)$$

Now we consider the connection between the $W_{1+\infty}$ algebra which includes the redundant variables of $\{t_{mp}\}$ and the algebra of \bar{W} which doesn't include $\{t_{mp}\}$.

Firstly, we give some notations here. From the expression of $P_{kp}^{(i)}$, we know that every $P_{kp}^{(i)}$ is a summation of items of normal product. For each $P_{kp}^{(i)}$, these items can be divided into two categories; one includes all items that comprise none of variables in $\{t_{mp} | m \in \mathbb{N}\}$ and the other includes all items that comprise at least one variable in $\{t_{mp}\}$. We represent the summation of all items in the former category by $\bar{P}_{kp}^{(i)}$ and the summation of all items in the latter category by $\hat{P}_{kp}^{(i)}$. So we have $P_{kp}^{(i)} = \bar{P}_{kp}^{(i)} + \hat{P}_{kp}^{(i)}$ and $\bar{P}_{kp}^{(i)} = P_{kp}^{(i)}|_{t_{mp}=0}$ for every $P_{kp}^{(i)}$. Similarly we denote $P_{kp}^{(i)}|_{\frac{\partial}{\partial m_p}=0}$ and the other part by $\tilde{P}_{kp}^{(i)}$ and $\check{P}_{kp}^{(i)}$ respectively. And these notations are also applicable to $\{W_{kp}^{(i)}\}$.

Now we give the connection between the algebra of $W_{kp}^{(i)}$ and the algebra of $\bar{W}_{kp}^{(i)}$.

Lemma 3.3. Expand $W_{kp}^{(n)}$ in $\{P_{-mp} | m \in \mathbb{N}\}$; then the coefficient of $P_{-m_1 p} \cdot P_{-m_2 p} \cdot \dots \cdot P_{-m_i p}$ are linear combination of $\bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(l)}$, $l = 1, \dots, n$. That is

$$\begin{aligned} W_{kp}^{(n)} = & \bar{W}_{kp}^{(n)} + \sum_{i=1}^n \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} P_{-m_1 p} P_{-m_2 p} \cdot \dots \cdot P_{-m_i p} (\text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(n-i)} \\ & + \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(n-i-1)} + \dots + \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(0)}), \end{aligned} \quad (3.11)$$

where the constants only depend on p and especially the constant in front of $P_{-m_i p} \cdot \bar{W}_{(m_i+k)p}^{(n-1)}$ equals n .

Proof: First we prove a more general conclusion, the expansion of $\{W_n^{(m)} | n \in \mathbb{Z}\}$ in P_{-mp} . Due to the normal product, we have

$$\sum_{n=-\infty}^{\infty} \lambda^{-n-m} W_n^{(m)} \quad (3.12)$$

$$= \partial_{\mu}^m |_{\mu=\lambda} : \exp \sum_{i=-\infty}^{\infty} \left(\frac{P_i}{i\lambda^i} - \frac{P_i}{i\mu^i} \right) : \quad (3.13)$$

$$= \partial_\mu^m|_{\mu=\lambda} \left(\exp \sum_{i=1}^{\infty} \frac{P_{-ip} \cdot \mu^{ip}}{ip} : \exp \left(\sum_{i=-\infty}^{\infty} \frac{P_i}{i\lambda^i} - \sum_{\substack{i=-\infty \\ i \neq -kp}}^{\infty} \frac{P_i}{i\mu^i} \right) : \right) \quad (3.14)$$

$$= \sum_{l=0}^m C_m^l \cdot \partial_\mu^l|_{\mu=\lambda} \left(\exp \sum_{i=1}^{\infty} \frac{P_{-ip} \cdot \mu^{ip}}{ip} \right) \cdot C_m^{m-l} \partial_\mu^{m-l}|_{\substack{\mu=\lambda \\ t_{kp}=0}} \left(: \exp \sum_{i=-\infty}^{\infty} \left(\frac{P_i}{i\lambda^i} - \frac{P_i}{i\mu^i} \right) : \right) \quad (3.15)$$

$$= \sum_{l=0}^m \left(C_m^l \sum_{i=0}^l \sum_{m_i \in \mathbb{N}} \text{constant} \cdot \left(\prod_i P_{-m_i p} \right) \cdot \lambda^{(m_1+m_2+\dots+m_i)p-l} \right) \left(C_m^{m-l} \sum_{n=-\infty}^{\infty} \lambda^{-n-(m-l)} \bar{W}_n^{(m-l)} \right). \quad (3.16)$$

For any fixed n and m , compare the coefficient of λ^{-n-m} on both sides. Then we have

$$W_n^{(m)} = \bar{W}_n^{(m)} + m \cdot \sum_{m_1 \in \mathbb{N}} P_{-m_1 p} (\bar{W}_{m_1 p+n}^{(m-1)} + \text{constant} \cdot \bar{W}_{m_1 p+n}^{(m-2)} + \dots + \text{constant} \cdot \bar{W}_{m_1 p+n}^{(0)}) \quad (3.17)$$

$$+ \sum_{i=2}^n \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} P_{-m_1 p} P_{-m_2 p} \cdot \dots \cdot P_{-m_i p} (\text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i)p+n}^{(m-i)} + \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i)p+n}^{(m-1-i)} + \dots + \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i)p+n}^{(0)}) \quad (3.18)$$

Then, let $n = kp$ and we obtain that the coefficient of $P_{-m_1 p} \cdot P_{-m_2 p} \cdot \dots \cdot P_{-m_i p}$ are linear combination of $\bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(l)}$, $l = 1, \dots, n-1$.

Meanwhile, the constants in (3.11) are linear combination of the constants in the first bracket in (3.15). Since the latter are only depend on p when $n = kp$, we obtain that the constants in (3.11) are only depend on p . □

Now we extend the above conclusion to the algebra of $\{P_{kp}^{(n)}\}$, which is used in next section to prove our main theorem.

Theorem 3.4.

$$P_{kp}^{(n)} = \bar{P}_{kp}^{(n)} + \sum_{i=1}^n \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} P_{-m_1 p} P_{-m_2 p} \cdot \dots \cdot P_{-m_i p} (\text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_i+k)p}^{(n-i)} + \text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_i+k)p}^{(n-1-i)} + \dots + \text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_i+k)p}^{(0)}) \quad (3.19)$$

$$+ \text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_i+k)p}^{(n-1-i)} + \dots + \text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_i+k)p}^{(0)} \quad (3.20)$$

where the constants only depend on p and especially the constant in front of $P_{-m_i p} \cdot \bar{P}_{(m_i+k)p}^{(n-1)}$ equals n .

Proof: From the expression of $W_{kp}^{(n)}$ and $P_{kp}^{(n)}$, we know that one could be regard as a linear representation of the other. That is,

$$W_{kp}^{(n)} = P_{kp}^{(n)} + \text{constant} \cdot P_{kp}^{(n-1)} + \dots + \text{constant} \cdot P_{kp}^{(1)}. \quad (3.21)$$

and

$$P_{kp}^{(n)} = W_{kp}^{(n)} + \text{constant} \cdot W_{kp}^{(n-1)} + \dots + \text{constant} \cdot W_{kp}^{(1)}, \quad (3.22)$$

where the constants only depend on kp . Use Lemma 3.3 and substitute the expansion of $W_{kp}^{(n-i)}$ in $\{P_{-mp}\}$ into (3.22). Then, we obtain

$$P_{kp}^{(n)} = \bar{W}_{kp}^{(n)} + \text{constant} \cdot \bar{W}_{kp}^{(n-1)} + \dots + \text{constant} \cdot \bar{W}_{kp}^{(2)} \quad (3.23)$$

$$+ \sum_{i=1}^n \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} P_{-m_1 p} P_{-m_2 p} \cdot \dots \cdot P_{-m_i p} (\text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(n-i)} \quad (3.24)$$

$$+ \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(n-1-i)} + \dots + \text{constant} \cdot \bar{W}_{(m_1+m_2+\dots+m_i+k)p}^{(0)}). \quad (3.25)$$

Let $t_{mp} = 0$ on both sides of (3.21) and (3.22) and we obtain

$$\bar{W}_{kp}^{(n)} = \bar{P}_{kp}^{(n)} + \text{constant} \cdot \bar{P}_{kp}^{(n-1)} + \dots + \text{constant} \cdot \bar{P}_{kp}^{(1)}, \quad (3.26)$$

and

$$\bar{P}_{kp}^{(n)} = \bar{W}_{kp}^{(n)} + \text{constant} \cdot \bar{W}_{kp}^{(n-1)} + \dots + \text{constant} \cdot \bar{W}_{kp}^{(1)}. \quad (3.27)$$

Substitute (3.26) with $n = 2, 3, \dots, n-1$ and (3.27) into (3.25). Then we obtain the expansion of (3.20), in which the constants are different from (3.25), but they are still depend only on p . Meanwhile, the item $P_{m_i p} \cdot P_{(m_i+k)p}^{(n-1)}$ is generated only by the item of $W_{kp}^{(n)}$. And the constant in front of it equals the constant in front of $P_{-m_i p} \cdot W_{(m_i+k)p}^{(n-1)}$. So the constant in front of $P_{-m_i p} \cdot P_{(m_i+k)p}^{(n-1)}$ also equals n . □

In next section, we'll give the expansions of $P_{kp}^{(3)}$, $P_{kp}^{(4)}$ and $P_{kp}^{(5)}$, which can be regard as examples and verifications of Theorem 3.4 with $n = 3, 4, 5$.

These conclusions also hold for $\{\tilde{W}_{kp}^{(n)}\}$ and $\{\tilde{P}_{kp}^{(n)}\}$, that is

Corollary 3.5.

$$P_{kp}^{(n)} = \tilde{P}_{kp}^{(n)} + \sum_{i=1}^n \sum_{\substack{m_1, m_2, \dots, m_i; \\ n_1, n_2, \dots, n_i \in \mathbb{N}}} P_{-m_1 p} P_{-m_2 p} \cdot \dots \cdot P_{-m_i p} (\text{constant} \cdot \tilde{P}_{(m_1+m_2+\dots+m_i+k)p}^{(n-i)} + \quad (3.28)$$

$$\text{constant} \cdot \tilde{P}_{(m_1+m_2+\dots+m_i+k)p}^{(n-1-i)} + \dots + \text{constant} \cdot \tilde{P}_{(m_1+m_2+\dots+m_i+k)p}^{(0)}) P_{n_1 p} P_{n_2 p} \cdot \dots \cdot P_{n_i p} \quad (3.29)$$

where the constants only depend on p .

Similarly, there is the same connection between $W_{kp}^{(n)}$ and $\tilde{W}_{kp}^{(n)}$.

Based on this connection, we can calculate the commutation relations of one algebra from those of another algebra. As an example, we show the calculation for $n = 2$. The conclusion through the method is the same as those through a straightforward calculation; but the calculation is much simpler. This can also be regard as a verification of Theorem 3.4 with $n = 2$.

Corollary 3.6. $\{\frac{1}{2}\bar{P}_{kp}^{(2)}\}$ has the same commutation relations as $\{\frac{1}{2}P_{kp}^{(2)}\}$.

Proof: $\{\frac{1}{2}P_{kp}^{(2)}|k = -1, 0, 1, \dots\}$ is a subalgebra of the Virasoro algebra. Then their commutation relations are

$$[\frac{1}{2}P_{np}^{(2)}, \frac{1}{2}P_{mp}^{(2)}] = \frac{1}{2}(np - mp)P_{np+mp}^{(2)} + \frac{1}{12}((np)^3 - np)\delta_{np+mp,0}, \quad (3.30)$$

where $m, n \in \{-1, 0, 1, \dots\}$. By direct computation, we have

$$P_{kp}^{(2)} = \bar{P}_{kp}^{(2)} + 2 \cdot \sum_{m \in \mathbb{N}} P_{-mp} P_{(k+m)p}. \quad (3.31)$$

Substitute it into (3.30) and note that $\bar{P}_{kp}^{(2)}$ don't include the variables of $\{t_{mp}\}$ so that they are commutable with P_{kp} . So we have

$$\frac{1}{2}(np - mp)\bar{P}_{np+mp}^{(2)} + (n - m)p \cdot \sum_{l_3 \in \mathbb{N}} P_{-l_3p} P_{(l_3+n+m)p} + \frac{1}{12}((np)^3 - np)\delta_{np+mp,0} \quad (3.32)$$

$$= [\frac{1}{2}\bar{P}_{np}^{(2)} + \sum_{l_1 \in \mathbb{N}} P_{-l_1p} P_{(l_1+n)p}, \frac{1}{2}\bar{P}_{mp}^{(2)} + \sum_{l_2 \in \mathbb{N}} P_{-l_2p} P_{(l_2+m)p}] \quad (3.33)$$

$$= [\frac{1}{2}\bar{P}_{np}^{(2)}, \frac{1}{2}\bar{P}_{mp}^{(2)}] + [\sum_{l_1 \in \mathbb{N}} P_{-l_1p} P_{(l_1+n)p}, \sum_{l_2 \in \mathbb{N}} P_{-l_2p} P_{(l_2+m)p}] \quad (3.34)$$

$$= [\frac{1}{2}\bar{P}_{np}^{(2)}, \frac{1}{2}\bar{P}_{mp}^{(2)}] + \sum_{n+l_1=l_2} P_{-l_1p} \cdot (n + l_1)p \cdot P_{(m+l_2)p} - \sum_{m+l_2=l_1} P_{-l_2p} \cdot (m + l_2)p \cdot P_{(n+l_1)p} \quad (3.35)$$

$$= [\frac{1}{2}\bar{P}_{np}^{(2)}, \frac{1}{2}\bar{P}_{mp}^{(2)}] + \sum_{l_1 \in \mathbb{N}} P_{-l_1p} \cdot (n - m)p \cdot P_{(m+n+l_1)p} \quad (3.36)$$

Cancel the same item on both sides and we have

$$[\frac{1}{2}P_{np}|_{t_{lp}=0}^{(2)}, \frac{1}{2}P_{mp}|_{t_{lp}=0}^{(2)}] = [\frac{1}{2}P_{np}^{(2)}, \frac{1}{2}P_{mp}^{(2)}]|_{t_{lp}=0} \quad (3.37)$$

$$= \frac{1}{2}(np - mp)P_{np+mp}^{(2)} + \frac{1}{12}((np)^3 - np)\delta_{np+mp,0}; \quad l \in \mathbb{N} \quad (3.38)$$

which is the same as $\{\frac{1}{2}P_{kp}^{(2)}\}$ □

§4 Precise constraints on a associated tau function

In this section, based on Theorem 3.4 we propose a new and computable method which can determine the values of unknown constants in higher order constraints on a $\tau(t)$ function which satisfies both the p-reduced KP hierarchy and the sting equation. Consequently, with these values we can obtain a precise algebra that the higher constraints compose. Meanwhile, the algebra includes the Virasoro algebra as its subalgebra. So, the conclusion in [5, 6], which is

that a τ function under the constraints of 2-reduced KP and the string equation is a vacuum vector of the Virasoro algebra, is also included in our conclusion.

Now we first give a preliminary conclusion which shows how the unknown constants in constraints on a corresponding τ function generates. Similar result was obtained through other method. Here we use a simple way to obtain it.

Lemma 4.7. When $k = -1, 0, 1, \dots; i \in \mathbb{N}$

$$\bar{P}_{kp}^{(i)} \cdot \tau(t) = c_k^{(i)} \cdot \tau(t), \quad (4.1)$$

where each $c_k^{(i)}$ is a constant.

Proof: From (3.10), we obtain

$$W_{kp}^{(j+1)} \cdot \tau(t) = \text{constant} \cdot \tau(t) \quad \text{when} \quad k = 0, 1, 2, \dots; \quad j = 1, 2, \dots. \quad (4.2)$$

Note a fact that

$$-\prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) (G(z) - 1) \left(\frac{P_{-p} \cdot \tau(t)}{\tau(t)} \right) = \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) z^{-p}. \quad (4.3)$$

Add (4.3) to (3.9) and we obtain

$$(G(z) - 1) \frac{\left(\frac{W_{kp}^{(j+1)}}{j+1} - \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) t_p \right) \cdot \tau(t)}{\tau(t)} = 0 \quad \text{when} \quad k = -1; \quad j \in \mathbb{N}. \quad (4.4)$$

That is

$$\left(W_{kp}^{(j+1)} - (j+1) \prod_{r=0}^{j-1} \left(\frac{p-1}{2} - r \right) t_p \right) \cdot \tau(t) = \text{constant} \cdot \tau(t) \quad \text{when} \quad k = -1; \quad j \in \mathbb{N}. \quad (4.5)$$

Consider (4.2) and (4.5) together. Let $t_{mp} = 0$ on both sides of the two identities. We could cancel the items that include variables of $\{t_{mp}\}$. Since the $\tau(t)$ of a p-reduced KP is independent on the variables of $\{t_{mp}\}$, we obtain

$$\bar{W}_{kp}^{(j+1)} \cdot \tau(t) = \text{constant} \cdot \tau(t) \quad \text{when} \quad k = -1, 0, 1, 2, \dots; \quad j \in \mathbb{N}. \quad (4.6)$$

Meanwhile, we have

$$P_{kp}^{(n)} = W_{kp}^{(n)} + \text{constant} \cdot W_{kp}^{(n-1)} + \dots + \text{constant} \cdot W_{kp}^{(1)}. \quad (4.7)$$

Let $t_{mp} = 0$ on both sides, and we have

$$\bar{P}_{kp}^{(n)} = \bar{W}_{kp}^{(n)} + \text{constant} \cdot \bar{W}_{kp}^{(n-1)} + \dots + \text{constant} \cdot \bar{W}_{kp}^{(1)}. \quad (4.8)$$

So, together with (4.6) and that the τ function is independent on t_{mp} , we obtain

$$\bar{P}_{kp}^{(n)} \cdot \tau(t) = \text{constant} \cdot \tau(t) \quad k = -1, 0, 1, \dots; i \in \mathbb{N} \quad (4.9)$$

□

From the above lemma, we know that there are a lot of unknown constants, $c_k^{(i)}$, in the constraints. Now, we propose a method to determine the values of these constants and obtain the precise constraints.

Theorem 4.8. The constants $c_k^{(i)}$, $k = 0, 1, 2, \dots, i \in \mathbb{N}$ can be determined by a recursive process step by step, in which

$$c_k^{(i)} = 0, k = 1, 2, 3, \dots \quad (4.10)$$

and

$$c_0^{(1)}(p) = 0, \quad c_0^{(2)}(p) = -\frac{1}{12}(p^2 - 1), \quad c_0^{(3)}(p) = 0, \quad c_0^{(4)}(p) = -\left(\frac{7}{240}p^2 - \frac{1}{80}\right)(p^2 - 1) \cdots \quad (4.11)$$

Proof: This method is essentially a recursive process. Here we use the second mathematical reduction on i to prove it. Start from $i = 2$. Let $k = -1$ and $j = 2$ in (3.9), and we have

$$(G(z) - 1) \frac{\frac{1}{3}(P_{-p}^{(3)} - \frac{3}{2}(-p+2)P_{-p}^{(2)} + (-p+1)(-p+2)P_{-p}) \cdot \tau(t)}{\tau(t)} = -\frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}. \quad (4.12)$$

Applying Lemma 3.4 with $n = 2, 3$ and by a straightforward computation, we have

$$P_{-p}^{(3)} \cdot \tau(t) = \bar{P}_{-p}^{(3)} \cdot \tau(t) + \left(\sum_{m=1}^{\infty} 3 \cdot P_{-mp} \cdot \bar{P}_{(m-1)p}^{(2)} \right) \cdot \tau(t) + \left(\sum_{n,l \in \mathbb{N}} 3 \cdot P_{-np} \cdot P_{-lp} \cdot P_{(n+l-1)p} \right) \cdot \tau(t). \quad (4.13)$$

Here the third item in the right-hand side of the above identity equals 0 since $n + l - 1 \geq 0$ and $\tau(t)$ is independent on t_{mp} ; using Lemma 4.7, the first two items in the right-hand side equal $c_{-1}^{(3)} \cdot \tau(t) + \left(\sum_{m=1}^{\infty} 3 \cdot P_{-mp} \cdot c_{(m-1)p}^{(2)} \right) \cdot \tau(t)$. As for $P_{-p}^{(2)}$, the conclusion is similar. And we have

$$P_{-p}^{(2)} \cdot \tau(t) = \bar{P}_{-p}^{(2)} \cdot \tau(t) + \sum_{n=1}^{\infty} 2 \cdot P_{-np} \cdot P_{(n-1)p} \tau(t) = c_{-1}^{(2)} \cdot \tau(t) \quad (4.14)$$

Substitute (4.13) and (4.14) into (4.12) and we have

$$(G(z) - 1) \left(\sum_{m=1}^{\infty} P_{-mp} \cdot c_{(m-1)p}^{(2)} + \frac{1}{3}(-p+1)(-p+2)P_{-p} \right) = -\frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}. \quad (4.15)$$

By straightforward computation, it is

$$\sum_{m=1}^{\infty} z^{-mp} \cdot c_{(m-1)p}^{(2)} + \frac{1}{3}(-p+1)(-p+2)z^{-p} = \frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}. \quad (4.16)$$

Comparing the coefficient of z^{-mp} on both sides, we obtain

$$\begin{cases} c_0^{(2)} = -\frac{1}{12}(p^2 - 1), \\ c_k^{(2)} = 0; \quad k = 1, 2, 3, \dots \end{cases} \quad (4.17)$$

So the theorem holds when $i = 2$.

Assume the theorem holds for $i \leq n - 2$, that is $c_k^{(i)} = 0$ where $k \in \mathbb{N}$ and $c_0^{(i)}$ are already determined.

Now we prove the theorem holds for $i = n - 1$. From (3.9) with $j = n - 1$, we have

$$(G(z) - 1) \frac{\frac{W_{-p}^{(n)}}{n} \cdot \tau(t)}{\tau(t)} = - \prod_{r=0}^{n-2} \left(\frac{p-1}{2} - r \right) z^{-p}, \quad (4.18)$$

in which

$$W_{-p}^{(n)} \cdot \tau(t) = (P_{-p}^{(n)} + \text{constant} \cdot P_{-p}^{(n-1)} + \dots + \text{constant} \cdot P_{-p}^{(1)}) \cdot \tau(t). \quad (4.19)$$

Applying Theorem 3.4 with $k = -1$. Because $\sum_i -m_i \neq -1$ when $i \geq 2$ and $P_{-p}^{(j)}$ doesn't include a single item of $P_{-mp}^{(1)}$ when $j \geq 2$, we can cancel some constants in the expansion of $P_{-p}^{(j)}$. Then each $P_{-p}^{(j)}$ with $j \geq 2$ in the above identity can be expanded as

$$P_{-p}^{(j)} = \bar{P}_{-p}^{(j)} + j \cdot \sum_{m=1}^{\infty} P_{-mp} \left(\sum_{k=1}^{j-1} \bar{P}_{(m-1)p}^{(k)} \right) \quad (4.20)$$

$$+ \sum_{m_1, m_2 \in \mathbb{N}} P_{-m_1 p} P_{-m_2 p} \left(\sum_{k=1}^{j-2} \text{constant} \cdot \bar{P}_{(m_1+m_2-1)p}^{(k)} \right) \quad (4.21)$$

$$+ \dots \quad (4.22)$$

$$+ \sum_{m_1, m_2, \dots, m_{j-1}} P_{-m_1 p} P_{-m_2 p} \dots P_{-m_{j-1} p} (\text{constant} \cdot \bar{P}_{(m_1+m_2+\dots+m_{j-1}-1)p}^{(1)}) \quad (4.23)$$

Substitute the above into (4.18) and we obtain

$$(G(z) - 1) \frac{A}{n\tau(t)} = - \prod_{r=0}^{n-2} \left(\frac{p-1}{2} - r \right) z^{-p} \quad (4.24)$$

where

$$A = \sum_{m \in \mathbb{N}} P_{-mp} \left(n \cdot \sum_{k=1}^{n-1} c_{m-1}^{(k)} \right) + \sum_{i=2}^{n-1} \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} \prod_{l=1}^i P_{-m_l p} \left(\sum_{k=1}^{n-i} \text{constant} \cdot c_{m_1+m_2+\dots+m_i-1}^{(k)} \right) \quad (4.25)$$

$$+ \text{constant} \cdot \left(\sum_{i=1}^{n-2} \sum_{m_1, m_2, \dots, m_i \in \mathbb{N}} \prod_{l=1}^i P_{-m_l p} \left(\sum_{k=1}^{n-1-i} \text{constant} \cdot c_{m_1+m_2+\dots+m_{i-1}}^{(k)} \right) \right) \quad (4.26)$$

$$+ \dots \quad (4.27)$$

$$+ \text{constant} \cdot P_{-p}^{(1)} \quad (4.28)$$

Substitute the results of $i \leq n-2$ into the above, we obtain

$$\begin{aligned} (G(z) - 1) & \frac{\left(\sum_{m \in \mathbb{N}} P_{-mp} \cdot c_{m-1}^{(n-1)} + \sum_{l=2}^{n-1} \text{constant} \cdot P_{-p} \cdot c_0^{k-l} + \text{constant} \cdot P_{-p} \right) \cdot \tau(t)}{\tau(t)} \\ & = - \prod_{r=0}^{n-2} \left(\frac{p-1}{2} - r \right) z^{-p} \end{aligned} \quad (4.29)$$

Comparing the coefficients of z^{mp} on both sides, we obtain

$$\begin{cases} c_0^{(n-1)} + \sum_{l=2}^{n-1} \text{constant} \cdot c_0^{k-l} + \text{constant} = - \prod_{r=0}^{n-2} \left(\frac{p-1}{2} - r \right), \\ c_k^{(n-1)} = 0; \quad k = 1, 2, 3, \dots \end{cases} \quad (4.30)$$

There is only one unknown variable, $c_0^{(n-1)}$, in the first equation. Then we can determined it from that. So $c_k^{n-1} = 0$ where $k \in \mathbb{N}$ and $c_0^{(n-1)}$ is also determined, that is, the theorem holds for $i = n-1$. According to the second mathematical reduction, the theorem holds. The proof also give us a recursive method which can calculate the $c_k^{(i)}$ step by step.

□

From the above proof, we know that the method is essentially a recursive process, that is, we could determine the constants step by step. When $p = 2$, $c_0^{(2)} = -\frac{1}{12}(p^2 - 1)$ through this method, which is the same as $c_0^{(2)}$ obtained through a direct computation of commutation relations of Virasoro algebra in Corollary 3.6. Moreover, it is usually that assign 0 to all $c_k^{(i)}$, but here, using this method, we know that all of them do not equal 0. The constants which do not equal 0 are close related to the centers of the algebra that the constraints constitute. As for $c_{-1}^{(i)}$, the are unknown; but $c_{-1}^{(2)} = 0$, which can obtained by the commutation relations of the Virasoro algebra.

Now, with these values of the constants, we could obtain the precise algebra of the constraints. Note that the above theorem is still holds if we substitute $\{\tilde{P}_{kp}^{(i)}\}$ for $\{P_{kp}^{(i)}\}$. So, we have

Corollary 4.9. If L satisfies the p-reduced KP hierarchy and $\frac{p-1}{2}L^{-p} = (ML^{-p+1})_-$, then L satisfies the String equation and the τ function of L is a vacuum vector of the algebra \bar{P} or \tilde{P} where

$$P = \{\bar{P}_{-p}^{(i)} - c_{-1}^{(i)}, \bar{P}_0^{(i)} - c_0^{(i)}, \bar{P}_{kp}^{(i)} | i, k \in \mathbb{N}\}. \quad (4.31)$$

and

$$P = \{\tilde{P}_{-p}^{(i)} - c_{-1}^{(i)}, \tilde{P}_0^{(i)} - c_0^{(i)}, \tilde{P}_{kp}^{(i)} | i, k \in \mathbb{N}\}. \quad (4.32)$$

That is

$$\bar{P} \cdot \tau(t) = 0 \quad \text{and} \quad \tilde{P} \cdot \tau(t) = 0. \quad (4.33)$$

Corollary 4.10. The algebra P includes a Virasoro algebra with no center as its subalgebra, that is, $\{\bar{P}_{np}^{(2)} - c_n^{(2)}\}$.

Proof: Let

$$\frac{1}{2}(\bar{P}_{np}^{(2)} - c_n^{(2)}) = L_n, n = -1, 0, 1, 2, \dots \quad (4.34)$$

Applying Corollary 3.6, we have

$$[L_n, L_m] = (n - m)L_{n+m}. \quad (4.35)$$

□

So, the conclusion in [5, 6], which is that a τ function under the constraints of 2-reduced KP and the string equation is a vacuum vector of the Virasoro algebra, is also included in our conclusion in [5, 6]. That is, we extend the sufficiency of the above conclusion from 2-reduced KP and the Virasoro algebra to an arbitrary p-reduced KP and a algebra of \bar{P} or \tilde{P} .

Now we show the detailed calculation when $i = 4$ and 5, which together with the case of $i = 3$ can be used as examples of Theorem 4.8 and Theorem 3.4.

Consider the case of $i = j + 1 = 4$ in (3.9). We have

$$\begin{aligned} (G(z) - 1) & \frac{\frac{1}{4}(P_{-p}^{(4)} - (6 - 2p)P_{-p}^{(3)} + (2p^2 - 9p + 11)P_{-p}^{(2)} - (1 - p)(2 - p)(3 - p)P_{-p}) \cdot \tau(t)}{\tau(t)} \\ & = -\frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p} \end{aligned} \quad (4.36)$$

By a straightforward computation,

$$P_{-p}^{(4)} = \bar{P}_{-p}^{(4)} + 4 \cdot \sum_{m \in \mathbb{N}} P_{-mp} \cdot \bar{P}_{mp-p}^{(3)} + \sum_{m_1, m_2 \in \mathbb{N}} 6 \cdot P_{-m_1 p} \cdot P_{-m_2 p} \cdot P_{(m_1 + m_2 - 1)p}^{(2)} \quad (4.37)$$

$$+ \sum_{m \in \mathbb{N}} 2 \cdot mp(mp - p)P_{-mp}P_{mp-p} + \sum_{m_1, m_2, m_3 \in \mathbb{N}} 4 \cdot P_{-m_1 p}P_{-m_2 p}P_{-m_3 p}P_{(m_1 + m_2 + m_3 - 1)p}. \quad (4.38)$$

Substitute (4.13) and (4.38) into (4.36) and we have

$$\begin{aligned} (G(z) - 1) & \frac{\sum_{m \in \mathbb{N}} P_{-mp} \cdot c_{m-1}^{(3)} + \frac{3}{2}(p-3) \sum_{m \in \mathbb{N}} P_{-mp} \cdot c_{m-1}^{(2)} + \frac{1}{4}(p-1)(p-2)(p-3)P_{-p} \cdot \tau(t)}{\tau(t)} \\ & = -\frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}. \end{aligned} \quad (4.39)$$

Comparing the coefficient of z^{-mp} on both sides, we obtain

$$\begin{cases} c_0^{(3)} - \frac{(p^2-1)(p-3)}{8} + \frac{(p-1)(p-2)(p-3)}{4} = \frac{(p-1)(p-3)(p-5)}{8}, \\ c_{m-1}^{(3)} = 0, \quad m = 2, 3, 4, \dots \end{cases} \quad (4.40)$$

Solving them, we have

$$\begin{cases} c_0^{(3)} = 0 \\ c_{m-1}^{(3)} = 0, \quad m = 2, 3, 4, \dots \end{cases} \quad (4.41)$$

Consider the case of $i = j + 1 = 5$ in (3.9). we have

$$\begin{aligned} (G(z) - 1) &= \frac{\frac{1}{5} \cdot \left((P_{-p}^{(5)} - \frac{5}{2}(4-p)P_{-p}^{(4)} + (\frac{10}{3}p^2 - 20p + 35)P_{-p}^{(3)} - \right. \\ &\quad \left. (-\frac{5}{2}p^3 + 20p^2 - \frac{105}{2}p + 50)P_{-p}^{(2)} + (1-p)(2-p)(3-p)(4-p)P_n) \right) \cdot \tau(t)}{\tau(t)} \\ &= - \prod_{r=0}^3 \left(\frac{p-1}{2} - r \right) z^{-p} \end{aligned} \quad (4.42)$$

In order to obtain the expansion of $P_{-p}^{(5)}$ in $\{t_{mp}\}$, we first calculate the items in $P_{-p}^{(5)}$ which include only one variable in $\{t_{mp}\}$. Note that

$$2 \sum_{i+j=kp} : i P_i P_j : = \sum_{i+j=kp} : i P_i P_j : + \sum_{i+j=kp} j : P_j P_i : = \sum_{i+j=kp} kp : P_i P_j : \quad (4.43)$$

Let $\{t_{mp} | m \in \mathbb{N}\} = 0$ on both sides, and we have

$$2 \sum_{\substack{i+j=kp \\ i,j \neq -mp}} : i P_i P_j : = \sum_{\substack{i+j=kp \\ i,j \neq -mp}} : i P_i P_j : + \sum_{\substack{i+j=kp \\ i,j \neq -mp}} : j P_j P_i : = kp \cdot \bar{P}_{kp}^{(2)}. \quad (4.44)$$

So the items which include only one variables in t_{mp} in $P_{-p}^{(5)}$ are

$$\begin{aligned} & \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \sum_{\substack{i+j+k+l=mp-p \\ i,j,k,l \neq -mp}} : P_i P_j P_k P_l : - \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \sum_{\substack{i+j=mp-p \\ i,j \neq -mp}} : ij P_i P_j : \\ & - \sum_{m \in \mathbb{N}} 5 \cdot (-mp) \cdot P_{-mp} \sum_{\substack{i+j=mp-p \\ i,j \neq -mp}} : j P_i P_j : - \sum_{m \in \mathbb{M}} 5 \cdot (-mp) \cdot P_{-mp} \sum_{\substack{i+j=mp-p \\ i,j \neq -mp}} : i P_i P_k : \\ & = \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \sum_{\substack{i+j+k+l=mp-p \\ i,j,k,l \neq -mp}} : P_i P_j P_k P_l : - \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \sum_{\substack{i+j=mp-p \\ i,j \neq -mp}} : ij P_i P_j : \end{aligned}$$

$$\begin{aligned}
& + \sum_{m \in \mathbb{N}} 5mp \cdot P_{-mp} \sum_{\substack{i+j=mp-p \\ i,j \neq -mp}} (mp-p) : P_i P_j : \\
& = \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \cdot \bar{P}_{mp-p}^{(4)} + \sum_{m \in \mathbb{N}} 5mp(mp-p) P_{-mp} \bar{P}_{mp-p}^{(2)}.
\end{aligned} \tag{4.45}$$

Substitute the above into the expansion. Then it can be written as

$$P_{-p}^{(5)} = \bar{P}_{-p}^{(5)} + \sum_{m \in \mathbb{N}} 5 \cdot P_{-mp} \cdot \bar{P}_{mp-p}^{(4)} + \sum_{m \in \mathbb{N}} 5mp(mp-p) P_{-mp} \bar{P}_{mp-p}^{(2)} \tag{4.46}$$

$$+ \sum_{m_1, m_2 \in \mathbb{N}} 10 \cdot P_{-m_1 p} \cdot P_{-m_2 p} \cdot \bar{P}_{(m_1+m_2-1)p}^{(3)} \tag{4.47}$$

$$+ \sum_{m_1, m_2 \in \mathbb{N}} 5 \cdot (m_1 \cdot m_2 + m_1(m_1 + m_2 - p) + m_2(m_1 + m_2 - p)) P_{-m_1 p} P_{-m_2 p} P_{(m_1+m_2-1)p} \tag{4.48}$$

$$+ \sum_{m_1, m_2, m_3 \in \mathbb{N}} 10 \cdot P_{-m_1 p} P_{-m_2 p} P_{-m_3 p} \bar{P}_{(m_1+m_2+m_3-1)p}^{(2)} \tag{4.49}$$

$$+ \sum_{m_1, m_2, m_3, m_4 \in \mathbb{N}} 5 \cdot P_{-m_1 p} P_{-m_2 p} P_{-m_3 p} P_{-m_4 p} P_{(m_1+m_2+m_3+m_4-1)p}. \tag{4.50}$$

Substitute the above expansion into (4.42). Then we obtain

$$\begin{aligned}
& (G(z) - 1) \frac{\left(\sum_{m \in \mathbb{N}} P_{-mp} \cdot c_{m-1}^{(4)} + \sum_{m \in \mathbb{N}} m(m-1)p^2 P_{-mp} \cdot c_{m-1}^{(2)} - (8-2p) \sum_{m \in \mathbb{N}} P_{-mp} \cdot c_{m-1}^{(3)} \right)}{\tau(t)} \\
& = -\frac{p-7}{2} \frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} z^{-p}.
\end{aligned} \tag{4.51}$$

Comparing the coefficient of z^{-mp} on both sides, we obtain

$$\begin{cases} c_0^{(4)} - (\frac{1}{6}p^2 - p + \frac{7}{4})(p^2 - 1) + \frac{1}{5}(p-1)(p-2)(p-3)(p-4) = \frac{p-1}{2} \cdot \frac{p-3}{2} \cdot \frac{p-5}{2} \cdot \frac{p-7}{2}, \\ c_{m-1}^{(3)} = 0, \quad m = 2, 3, 4, \dots \end{cases} \tag{4.52}$$

Solving them, we have

$$\begin{cases} c_0^{(4)} = (\frac{7}{240}p^2 - \frac{1}{80})(p^2 - 1), \\ c_{m-1}^{(3)} = 0, \quad m = 2, 3, 4, \dots \end{cases} \tag{4.53}$$

□

§5 Conclusions

In this study, for an arbitrary p , we propose a new method which can determine the values of unknown constants in constraints on a tau function which satisfies both the p -reduced KP hierarchy and the string equation. It is a recursive process and through it we could directly calculate the constants step by step. By this method, we know that, unlike what people usually think of, all of them do not equal 0. When $p = 2$, our conclusion is the same as the current conclusion, that is, the constants determined through our method being the same as those determined through commutation relations of Virasoro. With these values we obtain the precise algebra that the constraints compose. Meanwhile, the algebra includes the Virasoro algebra as its subalgebra. So, the conclusion in [5, 6], which is that a τ function under the constraints of 2-reduced KP and the string equation is a vacuum vector of the Virasoro algebra, is also included in our conclusion. That is, we also extend the sufficiency of the above conclusion from 2-reduced KP and the Virasoro algebra to an arbitrary p -reduced KP and an algebra of \bar{P} or \tilde{P} .

In this process we also obtain the connection between the $W_{1+\infty}$ algebra which includes the redundant variables of $\{t_{mp}\}$ and the algebra of $\bar{W} = \{W_n^{(m)}|_{t_{mp}=0}\}$ which doesn't include $\{t_{mp}\}$. Through this connection, we could obtain commutation relations of one algebra from those of the other algebra. And the calculation is much simpler than a straightforward calculation.

This method is a general approach which is feasible to other integrable systems similar to the KP system, such as dKP hierarchy, BKP hierarchy, qKP hierarchy etc. In the near future, we will try them. And we will study further the algebraic structure of the algebra \bar{P} .

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